Last Time: Vector spaces

$V \leftarrow$ set of "vectors"

$+$, $\cdot$
addition, scalar mult.

① $u + v = v + u$

② $u + (v + w) = (u + v) + w$

③ there is a zero-vector 0_V w/ $0_V + v = v$

④ Additive inverses: each v has a $-v$ w/ $v + (-v) = 0_V$

⑤ $(a+b)\cdot v = a\cdot v + b\cdot v$

⑥ $a\cdot(u+v) = a\cdot u + a\cdot v$

⑦ $a\cdot(b\cdot v) = (ab)\cdot v$

⑧ $1\cdot v = v$

Examples: $\mathbb{R}^n$, $\mathcal{M}_{m,n}(\mathbb{R}) = \left\{\begin{matrix} m\times n \text{ matrices} \\ \text{w/ entries in } \mathbb{R}\end{matrix}\right\}$,

$\mathcal{P}_n(\mathbb{R}) = \left\{\begin{matrix}\text{degree} \leq n \text{ polynomials} \\ \text{w/ coefficients in } \mathbb{R}\end{matrix}\right\}$, + sporadic examples.

$\mathrm{Func}(S, \mathbb{R}) = \{\text{functions } S \to \mathbb{R}\}$ ← Important example!

Prop: Let V be a vector space w/ $v \in V$ and $c \in \mathbb{R}$.

① $0\cdot v = 0_V$ ② $-v = (-1)\cdot v$ ③ $c\cdot 0_V = 0_V$.

pf: Let V be a v.s. w/ $v \in V$ and $c \in \mathbb{R}$.

① $0\cdot v = (0+0)\cdot v = 0\cdot v + 0\cdot v$ so subtracting $0\cdot v$ from both sides yields $0_V = 0\cdot v$.

② $0_V = 0\cdot v = (1+(-1))\cdot v = 1\cdot v + (-1)\cdot v$

So $0_V = v + (-1)\cdot v$ and subtracting v from both sides yields $-v = (-1)\cdot v$.

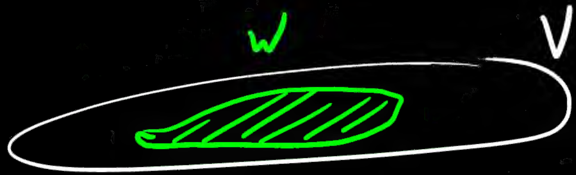
③ $c\cdot 0_V = c\cdot(0_V + 0_V) = c\cdot 0_V + c\cdot 0_V$, so subtracting $c\cdot 0_V$ from both sides yields $0_V = c\cdot 0_V$ $\blacksquare$

Subspaces

Idea: Find vector spaces within our vector spaces!

Def: Let V be a vector space. A **subspace** of V is a subset $W \subseteq V$ which is itself a vector space under the operations on V, restricted to W.

Unpacking This Definition:

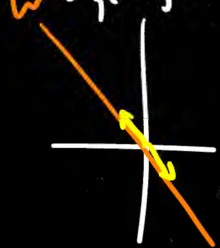


this "restricted operations" thing:

$$\begin{array}{cccccc} + : & V & \times & V & \to & V : (u,v)\mapsto u+v \\ & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ + : & W & \times & W & \to & W \end{array}$$

point: want addition of vectors in W to stay in W.

We also need scalar mult of vects in W to "stay" in W...



$$\begin{array}{cccccc} \cdot : & \mathbb{R} & \times & V & \to & V : (r,v)\mapsto r\cdot v \\ & & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \cdot : & \mathbb{R} & \times & W & \to & W \end{array}$$

Ex: Let $V = \mathbb{R}^3$ and $P = \{(x,y,z) \in \mathbb{R}^3 : x-y+3z=0\}$

Then P is a subspace of $\mathbb{R}^3$. To see this, we need to verify that P is a v.s. under the restricted operations from $\mathbb{R}^3$... ↙ Almost missed closure!

* ① ✓ (Comm): $+$ is comm on $\mathbb{R}^3$, it remains so in rest.
* ② ✓ (Assoc, +): $+$ is assoc on $\mathbb{R}^3$, so too on P.

③ (zero): We need to show $0_{\mathbb{R}^3} \in P$. Indeed:

* need! $(x,y,z) = (0,0,0)$ satisfies $0 = 0-0+3\cdot 0 = x-y+3z$.

Hence the zero-vector $(0,0,0) = 0_{\mathbb{R}^3} \in P$. ↑

Closure: Suppose $(x_1,x_2,x_3), (y_1,y_2,y_3) \in P$ and $c \in \mathbb{R}$.

* Need: $(x_1,x_2,x_3) + (y_1,y_2,y_3) \in P$ $\quad P+P \subseteq P$

and $c\cdot(x_1,x_2,x_3) \in P$.

Addition: $(x_1+y_1, x_2+y_2, x_3+y_3)$ needs to satisfy

* $$(x_1+y_1) - (x_2+y_2) + 3(x_3+y_3) \overset{?}{=} 0.$$

now $(x_1+y_1) - (x_2+y_2) + 3(x_3+y_3)$

$$= \underbrace{(x_1 - x_2 + 3x_3)} + \underbrace{(y_1 - y_2 + 3y_3)} \qquad x-y+3z=0$$

$$= 0 + 0 = 0 \quad \text{as desired.}$$

* Need Scalar Multiples: $c\cdot(x_1,x_2,x_3) = (cx_1, cx_2, cx_3)$ satisfies

$$cx_1 - cx_2 + 3cx_3 = c\underline{(x_1 - x_2 + 3x_3)} = c\cdot 0 = 0,$$

so $c\cdot(x_1,x_2,x_3) \in P$ as desired. *

Point: P is closed under $+$ and $\cdot$.

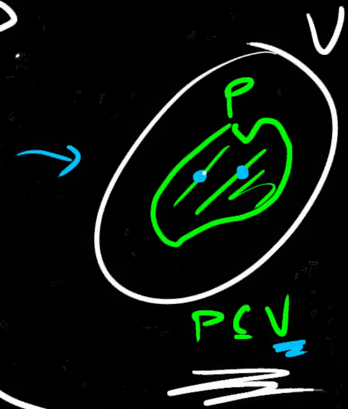
4 (Negatives): $(-1)\cdot v = -v$, so closure under scalar mult yields negatives as desired...

5 ("Left dist"): $a\cdot(u+v) \overset{?}{=} a\cdot u + a\cdot v$ in $\mathbb{R}^3$, so it's true in P.

6 ("Right dist"): $(a+b)\cdot v \overset{?}{=} a\cdot v + b\cdot v$ in $\mathbb{R}^3$, so it holds in P

7 ("assoc" for $\cdot$): $a\cdot(b\cdot v) \overset{?}{=} (ab)\cdot v$ in $\mathbb{R}^3$, so again in P!

8 ("Identity"): $1\cdot v = v$ so holds automatically in P.



Prop (Subspace Test): Let V be a vector space and let $S \subseteq V$. The following are equivalent.

1. S is a subspace of V.
2. S is closed under addition and scalar multiplication and $0_V \in S$.

NB: The proof was (in spirit) already done when we discussed $P \subseteq \mathbb{R}^3$ above.

Point of Subspace Test: If we want to show $S \subseteq V$ is a subspace of V, we only need to check three things: ① $0_V \in S$, ② S is closed under addition, ③ S is closed under scalar multiplication.

Ex: The trivial subspace of any vector space V is $\{0_V\} \subseteq V$. Let $S = \{0_V\}$. We know

① $0_V \in S$ ② $0_V + 0_V = 0_V$ so S closed under $+$

③ $c\cdot 0_V = 0_V$ so S is closed under scalar mult! $\boxtimes$

Ex: Let $S = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w=0\}$. Let's use the subspace test to show S is a subspace of $\mathbb{R}^4$.

① $0+0+0+0=0$ so $0_{\mathbb{R}^4} = (0,0,0,0) \in S$.

② Let $(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) \in S$.

Then $x_1 + y_1 + z_1 + w_1 = 0 = x_2 + y_2 + z_2 + w_2$.

Hence $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) + (w_1 + w_2)$

$$= (x_1 + y_1 + z_1 + w_1) + (x_2 + y_2 + z_2 + w_2)$$

$$= 0 + 0 = 0$$

Thus $(x_1, y_1, z_1, w_1) + (x_2, y_2, z_2, w_2) \in S$, and we see S is closed under vector addition!

③ Let $(x,y,z,w) \in S$ and $c \in \mathbb{R}$. Now

$x+y+z+w = 0$, so

$$cx + cy + cz + cw = c(x+y+z+w) = c \cdot 0 = 0$$

Hence $c \cdot (x,y,z,w) \in S$ and S is closed under scalar multiplication!

Hence S is a subspace of $\mathbb{R}^4$ by the subspace test! $\square$

<u>Notation</u>: We write "$S \leq V$" to mean "S is a subspace of V". That symbol is <u>NOT</u> the same as $S \subseteq V$ because these

$\uparrow$ subset

aren't the same concept!

Non-Ex: Let $S = \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathbb{R}^2 : x \in \mathbb{R} \right\}$.

$S \subseteq \mathbb{R}^2$ is a subset of $\mathbb{R}^2$.

But $S \not\leq \mathbb{R}^2$ (i.e. S is <u>not</u> a subspace of $\mathbb{R}^2$)

because... ① $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ x \end{pmatrix}$ for any x...

② $\begin{pmatrix} 1 \\ x \end{pmatrix} + \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ x+y \end{pmatrix} \neq \begin{pmatrix} 1 \\ z \end{pmatrix}$ for any z.

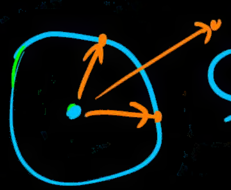
③ $c \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} c \\ cx \end{pmatrix} \in S$ if $c = 1$.

So S fails all three conditions... $\boxtimes$

Ex: The <u>trivial subspace</u> of any vector space V is $\{0_V\} \subseteq V$. Let $S = \{0_V\}$. We know

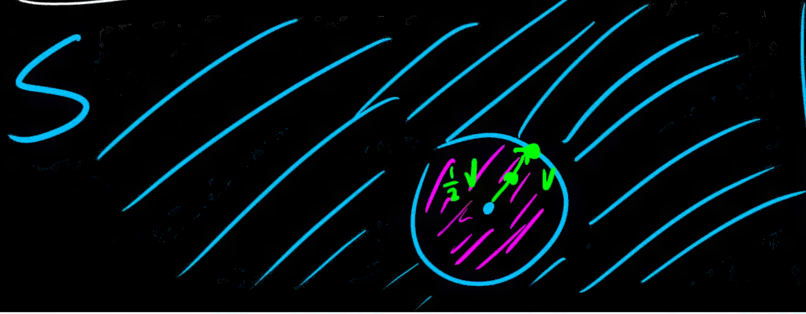
① $\boxed{0_V \in S}$ ② $0_V + 0_V = 0_V$ $\boxed{\text{so } S \text{ closed under } +}$

③ $c \cdot 0_V = 0_V$ so $\boxed{S \text{ is closed under scalar mult!}}$ $\boxtimes$



$\subseteq \mathbb{R}^2$

bad, b/c not closed under +.



$S \subseteq \mathbb{R}^2$

$0_V \in S$

$u, v \in S \Rightarrow u + v \in S$

S <u>NOT</u> closed under scaling.